

非线性弹性介质中冲击波 斜反射的研究 (II)

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摘要 与射前文结果求解了冲击波在固壁及滑动刚壁两种边界面上的斜反射问题。在滑动刚壁的情况下, 很可能出现马赫反射图象。

关键词 斜反射 固壁及滑动刚壁边界

一、引言

在上一篇文章中^[1], 我们给出了一种各向同性超弹性介质在平面应变下的本构方程、中心简单波解和平面冲击波解, 并求解了冲击波在自由面的斜反射问题。本文中我们利用文献[1]的结果求解了冲击波在固壁及滑动刚壁的斜反射问题。在滑动刚壁边界下, 当入射纵冲击波接近临界角时, 反射冲击波强度急剧增强, 预示着不规则反射中很可能出现马赫反射的图象。

二、基本公式

我们只列出所需的基本公式, 具体推导可见文献[1]。

(1) 本构方程

取坐标系 Ox_1x_2 如图1所示, 介质单位体积的内能 W 取为

$$W(p, q) = -2\mu q + \frac{1}{2}a \cdot p^2 + \frac{1}{3}dp^3 \quad (1)$$

μ, a, d 为材料常数, p, q 为平面应变张量的二个不变量

$$\left. \begin{aligned} p &= \epsilon_{11} + \epsilon_{22} \\ q &= \epsilon_{11}\epsilon_{22} - \epsilon_{12}\epsilon_{21} \end{aligned} \right\} \quad (2)$$

由 $\sigma_{ij} = \partial W / \partial \epsilon_{ij}$, 我们得到应力应变关系

$$\left. \begin{aligned} \sigma_{11} &= W_p + \epsilon_{22}W_q \\ \sigma_{22} &= W_p + \epsilon_{11}W_q \\ \sigma_{12} = \sigma_{21} &= -\epsilon_{21}W_q \end{aligned} \right\} \quad (3)$$

W 的下标表示对 p, q 的偏微分。

(2) 中心简单波解

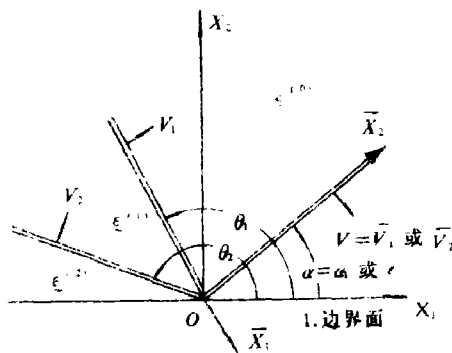


图1 斜反射图象

Fig. 1 Oblique reflected wave patterns
1—Boundary

有两族简单波

$$\left. \begin{aligned} \rho C_1^2 &= W_{pp} = a + 2dp \\ \rho C_2^2 &= -\frac{1}{2}W_q = \mu \end{aligned} \right\} \quad (4)$$

与 C_2 对应的为一冲击波解。与 C_1 对应的中心简单波解如下：其中应变、应力、速度各量均为波阵面与 X_1 轴交角 θ 的单变量函数。

应变

$$\left. \begin{aligned} \rho C_1^2 &= W_{pp} = a + 2dp = A^2 \cdot \sin^2 \theta \\ \varepsilon_{11} &= B_0 + (A^2/4d) \cdot \sin^4 \theta \\ \varepsilon_{22} &= -[(a/2d) + B_0] + (A^2/2d) \sin^2 \theta \cdot (2 - \sin^2 \theta) \\ \varepsilon_{12} = \varepsilon_{21} &= (A^2/32d)(\sin 4\theta - 4\theta + D_0) \end{aligned} \right\} \quad (5)$$

其中 A 、 B_0 、 D_0 为待定常数。

速度

$$\left. \begin{aligned} v_1 &= -8v^*(\sin^4 \theta - \beta_1) \\ v_2 &= v^*(4\theta - \sin 4\theta + \beta_2) \\ v^* &= (A^2/32d)(A^2/\rho)^{\frac{1}{2}} \end{aligned} \right\} \quad (6)$$

其中 β_1 、 β_2 为待定常数。

将 (5) 式代入本构关系 (3) 式中还可得到应力解。

(3) 平面冲击波解

平面冲击波的跳跃条件为

$$\left. \begin{aligned} [\sigma_{ia}]n_a + \rho V [v_i] &= 0 \\ [v_i]n_a + [v_a]n_i + 2V [\varepsilon_{ia}] &= 0 \end{aligned} \right\} \quad (7)$$

V 为冲击波法向波速， \underline{n} 为波阵面单位法向量。

如图 1 所示，取 \underline{n} 为 X_1 轴正向，我们得到两族冲击波，分别由 (8) 式和 (9) 式表示

$$[\bar{\varepsilon}_{11}] \neq 0, [\bar{\varepsilon}_{12}] = [\bar{\varepsilon}_{22}] = 0, \rho \bar{V}_1^2 = a + d(\bar{p}^+ + \bar{p}^-) \quad (8)$$

$$[\bar{\varepsilon}_{12}] \neq 0, [\bar{\varepsilon}_{11}] = [\bar{\varepsilon}_{22}] = 0, \rho \bar{V}_2^2 = -(1/2)W_q = \mu \quad (9)$$

其中 \bar{f} 表示 f 在 $OX_1 X_2$ 坐标系中的值， f^+ 、 f^- 分别为 f 在冲击波前、后方的值。

(8) 式表示纵向冲击波，(9) 式表示剪切冲击波。一般有 $V_1 > V_2$ ，所以 (8)、(9) 式也分别称为快冲击波和慢冲击波。

冲击波的稳定条件要求

$$C_i^+ \leq V_i \leq C_i^- \quad (10)$$

对于快冲击波，(10) 式要求 $d[p] \geq 0$ ，一般材料满足 $d < 0$ ，此时必须 $[p] < 0$ ，即为压缩冲击波。慢冲击波有 $V_2 = C_2 = \text{const}$ ，所以 V_2 波总是稳定的。

三、入射冲击波在固壁的斜反射

本文讨论的斜反射问题有两种图象，分别如图 1、2 所示。反射波一般由一族快波和一族慢波组成，其中慢波总是冲击波 V_2 ，而快波则为一族中心简单波 C_1 或冲击波 V_1 。

(1) 入射快冲击波 \bar{V}_1

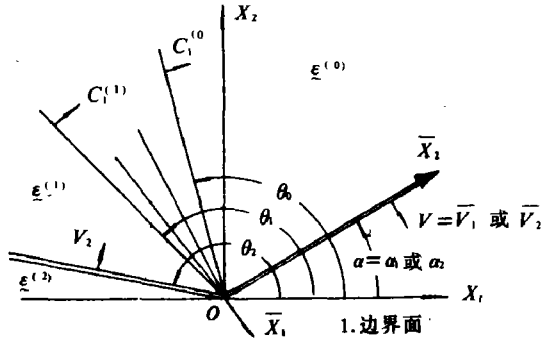


图2 斜反射图象

Fig. 2 Oblique reflected wave patterns
1—Boundary

n_1, n_2)。利用上节的冲击波解可以得到

$$\begin{aligned} \rho V_1^2 &= a + d(2\varepsilon_0 + \varepsilon_1) \\ \left\{ \begin{aligned} [\underline{v}]^{(1)} &= V_1 \varepsilon_1 \begin{pmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{pmatrix} \\ [\underline{v}]^{(2)} &= -2V_2 \gamma_2 \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} \end{aligned} \right. \end{aligned} \quad (14)$$

固壁边界要求

$$v^{(2)} = v^{(0)} + [\underline{v}]^{(1)} + [\underline{v}]^{(2)} = 0 \quad (15)$$

将(14)式代入(15)式得到

$$\left. \begin{aligned} V_1 \varepsilon_1 \cos(\theta_2 - \theta_1) + \bar{V}_1 \varepsilon_0 \cos(\theta_2 - \alpha_1) &= 0 \\ 2V_2 \gamma_2 \cos(\theta_2 - \theta_1) &= \bar{V}_1 \varepsilon_0 \sin(\theta_1 - \alpha_1) \end{aligned} \right\} \quad (16)$$

(16)式连同各族波有共同的视速度的条件

$$\bar{V}_1 / \sin\alpha_1 = V_1 / \sin\theta_1 = V_2 / \sin\theta_2 \quad (17)$$

即可解出 ε_1, θ_1 及 γ_2, θ_2 , 得到反射波解。

若反射波为简单波 C_1 和冲击波 V_2 。利用上节的中心简单波解, 我们可推出

$$\left. \begin{aligned} \sin^2\theta_2 / \sin^2\alpha_1 &= \mu / (a + d\varepsilon_0) \\ 8\text{tg}\theta_2 (\sin^4\theta_1 - \beta_1) + 4\theta_1 - \sin 4\theta_1 + \beta_2 &= 0 \\ \gamma_2 &= -\left(\frac{A^2}{8d}\right) \left(\frac{A^2}{\mu}\right)^{\frac{1}{2}} \cdot \frac{\sin^4\theta_1 - \beta_1}{\cos\theta_2} \end{aligned} \right\} \quad (18)$$

其中

$$\left. \begin{aligned} A^2 &= (a + d\varepsilon_0) / \sin^2\alpha_1 \\ \sin^2\theta_0 &= \sin^2\alpha_1 (a + 2d\varepsilon_0) / (a + d\varepsilon_0) \\ \beta_1 &= \sin^4\theta_0 - \left(\frac{a + d\varepsilon_0}{A^2}\right)^{\frac{1}{2}} \cdot \frac{4\varepsilon_0 d \sin\alpha_1}{A^2} \\ \beta_2 &= \sin 4\theta_0 - 4\theta_0 + (32d\varepsilon_0 \cos\alpha_1 / A^2) [(a + d\varepsilon_0) / A^2]^{\frac{1}{2}} \end{aligned} \right\} \quad (19)$$

入射波前假定为自然静止状态。

计算中取 $a/\mu = 2.5, d/\mu = -20$, 设入射波强度为 $[\bar{\varepsilon}_{11}]^{(0)} = \varepsilon_0 < 0$, 入射角 $\alpha = \alpha_1$, 则有

$$\rho \bar{V}_1^2 = a + d\varepsilon_0 \quad (11)$$

$$\underline{\varepsilon}^{(0)} = \varepsilon_0 \begin{pmatrix} \sin^2\alpha_1 & -\sin\alpha_1 \cos\alpha_1 \\ -\sin\alpha_1 \cos\alpha_1 & \cos^2\alpha_1 \end{pmatrix} \quad (12)$$

再由(7)式, 得到:

$$\underline{v}^{(0)} = \bar{V}_1 \varepsilon_0 \begin{pmatrix} -\sin\alpha_1 \\ \cos\alpha_1 \end{pmatrix} \quad (13)$$

先设反射波为冲击波 V_1 和 V_2 (图1)。并设 $[\bar{\varepsilon}_{11}]^{(1)} = \varepsilon_1 < 0, [\bar{\varepsilon}_{12}]^{(2)} = r_2 (\bar{X}_2$ 轴分别取 V_1, V_2 的阵面法向

由(18)、(19)式即可得到图 2 所示的反射波解。

图 3 给出了决定反射图象的 $\alpha_1 \sim \varepsilon_0$ 曲线。当入射冲击波的 $(\alpha_1, \varepsilon_0)$ 落在 AB 线以左时, 反射冲击波 V_1 和 V_2 ; $(\alpha_1, \varepsilon_0)$ 落在 AB 与 CD 线之间时, 反射简单波 C_1 和冲击波 V_2 ; $(\alpha_1, \varepsilon_0)$ 落在 CD 线以右时不再有规则反射解存在, 故 CD 线给出了临界角 α_c 与入射波强度 ε_0 的关系曲线。

AB 线为两种规则反射的分界线, $(\alpha_1, \varepsilon_0)$ 位于 AB 线上时, 只反射冲击波 V_2 。从 (16) 式不难看出, 此时入射波 \bar{V}_1 与反射波 V_2 的波阵面垂直。AB 线的解析式很容易推出为

$$\operatorname{tg} \alpha_1 = ((a + d\varepsilon_0)/\mu)^{\frac{1}{2}} \quad (20)$$

(2) 入射慢冲击波 \bar{V}_2

设入射波强度为 $(\bar{\varepsilon}_{12})^{(0)} = \gamma_0$, 入射角 $\alpha =$

α_2 。求解的过程与入射 \bar{V}_1 波时一样, 下面只列结果。4 - V_1 vanishes

反射 V_1 和 V_2 波时, 可由下式求解

$$\left. \begin{aligned} 2V_2\gamma_0 \sin 2\alpha_2 &= V_1 \varepsilon_1 \cos(\theta_1 + \alpha_2) \\ \gamma_2 \cos(\theta_1 + \alpha_2) &= \gamma_0 \cos(\theta_1 - \alpha_2) \\ V_1 / \sin \theta_1 &= V_2 / \sin \alpha_2 \\ \theta_2 &= \pi - \alpha_2 \\ \rho V_1^2 &= a + d\varepsilon_1 \end{aligned} \right\} \quad (21)$$

从 (21) 中第一式可看出, 只有 $\gamma_0 > 0$ 时, 才有 $\varepsilon_1 < 0$, 即反射快波为冲击波 V_1 。 $\gamma_2 = 0$ 时, 只反射冲击波 V_1 , 且有

$$\left. \begin{aligned} \theta_1 - \alpha_2 &= \pi/2 \\ \operatorname{tg}^2 \alpha_2 - (2\gamma_0 d/a) \operatorname{tg}^3 \alpha_2 &= \mu/a \end{aligned} \right\} \quad (22)$$

即此时反射波 V_1 与入射波 \bar{V}_2 的阵面垂直。(22)₂ 式给出图 4 中的 EE_1 线。

反射 C_1 和 V_2 波时, 可由下式求解

$$\left. \begin{aligned} 8\operatorname{tg} \alpha_2 (\sin^4 \theta_1 - \beta_1) - 4\theta_1 + \sin 4\theta_1 - \beta_2 &= 0 \\ r_2 &= \frac{A^2}{8d} \cdot \left(\frac{A^2}{\mu}\right)^{\frac{1}{2}} \cdot \frac{\sin^4 \theta_1 - \beta_1}{\cos \alpha_2} \end{aligned} \right\} \quad (23)$$

其中

$$\left. \begin{aligned} A^2 &= \mu / \sin^2 \alpha_2 \\ \sin^2 \theta_0 &= (a/\mu) \cdot \sin^2 \alpha_2 \\ \beta_1 &= \sin^4 \theta_0 - [(8\gamma_0 d \cos \alpha_2)/A^2](A^2/\mu)^{\frac{1}{2}} \\ \beta_2 &= \sin 4\theta_0 - 4\theta_0 - [(64\gamma_0 d \sin \alpha_2)/A^2](A^2/\mu)^{\frac{1}{2}} \end{aligned} \right\} \quad (24)$$

图 4 给出决定反射图象的 (α_2, γ_0) 曲线。 $\gamma_0 > 0$ 时反射 V_1 和 V_2 波, $\gamma_0 < 0$ 时反射 C_1 和 V_2 波; (α_2, γ_0) 位于 EF 线上时, 只反射快波; (α_2, γ_0) 位于 GH 之右时没有规则反射解。

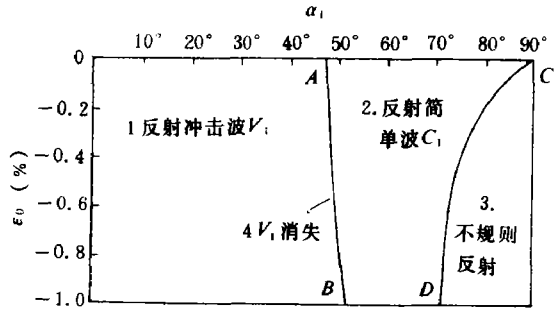


图 3 固壁入射 V_1 时的反射图象分区

Fig. 3 Reflection patterns of the incident wave V_1 on the rigid boundary

- 1—reflected shock wave V_1
- 2—reflected simple wave C_1
- 3—irregular reflection

4 - V_1 vanishes

四、入射冲击波在滑动刚壁的斜反射

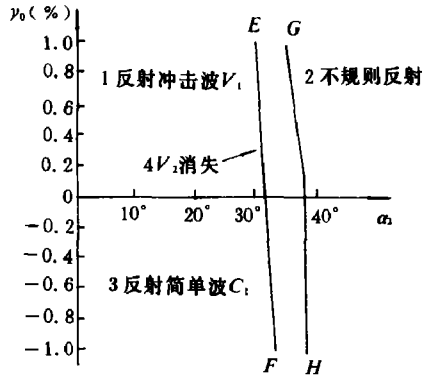


图4 固壁入射 V_2 时的反射图象分区

Fig. 4 Reflection patterns of the incident wave V_2 on the rigid boundary

- 1—Reflected shock wave V_1
- 2—Irregular reflection
- 3—Reflected simple wave C_1
- 4— V_2 vanishes

滑动刚壁边界条件为

$$\left. \begin{aligned} \sigma_{12}^{(2)} &= 0 \\ \sigma_2^{(2)} &= 0 \end{aligned} \right\} \quad (25)$$

(1) 入射快冲击波 \bar{V}_1

设反射波为冲击波 V_1 和 V_2 ,可以推出

$$\left. \begin{aligned} \gamma_2 &= 0 \\ \epsilon_0 \sin 2\alpha_1 + \epsilon_1 \sin 2\theta_1 &= 0 \\ V_1 / \sin \theta_1 &= \bar{V}_1 / \sin \alpha_1 \\ \rho \bar{V}_1^2 &= a + d \epsilon_0 \\ \rho V_1^2 &= a + d(2\epsilon_0 + \epsilon_1) \end{aligned} \right\} \quad (26)$$

(26)₁式表明不存在反射波 V_2 ;分析(26)式还可知恒有 $\epsilon_1 < 0$,所以反射波只有快冲击波 V_1 。

图5中 MN 线给出临界入射角 α_c 与入射波强度 ϵ_0 的关系曲线。图6绘出了反射波与入射波的强度比随入射角 α_1 的变化曲线,值得注意的是,当入射角趋近临界角时,反射波强度急剧增大,这预示着不规则反射很可能是类似于气体动力学中的Mach反射图象。并且,由于在本文中,此时只有一个反射波 V_1 ,很有利于求解不规则反射问题。

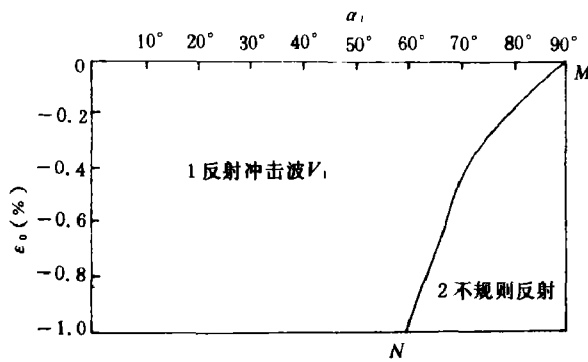


图5 滑动刚壁入射 V_1 时的反射图象分区

Fig. 5 Reflection patterns on the lubricated-rigid boundary

- 1—Reflected shock wave V_1
- 2—Irregular reflection

(2) 入射慢冲击波 \bar{V}_2

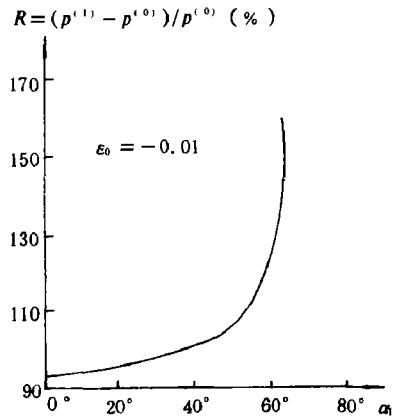


图6 滑动刚壁入射 V_1 时反射波与入射波强度之比

Fig. 6 The ratio of the amplitudes of the reflected and incident waves when incident wave V_1 on the lubricated-rigid boundary

此时可简单验证出只反射慢冲击波 V_2 ，且有：

$$\gamma_2 = -\gamma_0, \quad \theta_2 = \pi - \alpha_2 \quad (27)$$

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STUDIES ON OBLIQUE REFLECTION OF SHOCK WAVE IN NONLINEAR ELASTIC MEDIA (II)

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ABSTRACT In a former paper¹, the problem of oblique reflection of shock wave in isotropic hyperelastic media under plane-strain condition was discussed by using small deformation theory. The constitutive equations of materials and the solutions for simple waves and shock waves were presented. An solution for oblique reflection of a pressure shock wave impinging on a free surface was given.

In this paper, the constitutive equations and the solutions for simple waves and shock waves developed in the former paper are presented first, and then the oblique reflection of shock waves impinging on a rigid boundary and a lubricated-rigid boundary is studied.

The reflected wave patterns are shown in Fig. 1 and Fig. 2. For the medium we considered here, one of the reflected waves (slow wave) is always a shock wave, and the other reflected wave (fast wave) may be either a shock wave or a simple wave. In the case of rigid boundary, the reflected fast wave changes from a shock wave to a simple wave as the incident angle of the incident pressure shock wave increases. The reflected fast wave will vanish when the incident wave satisfies some certain conditions. Fig. 3 shows the correspondence of the reflected wave patterns with the incident wave conditions. Fig. 4 presents the results for incident shear shock waves. In this case, the reflected fast wave is a shock wave or a simple wave depending upon the sign of the incident shear shock wave. Again, for some certain incident shear shock waves, there exists only one reflected wave. In the case of lubricated-rigid boundary, a incident pressure shock wave generates only a reflected pressure shock wave, as shown in Fig. 5, and a incident shear shock wave generates an identical shear shock wave. Fig. 6 shows the ratio between the amplitudes of the reflected and the incident pressure shock waves. It is noticeable that when the incident angle tends to the critical angle, the amplitude of the reflected shock wave increases. we think that a Mach reflection pattern similar to that in gas dynamics is very likely to take place in irregular reflection.

KEY WORDS oblique reflection, rigid and lubricated-rigid boundary.